

Reviving the Lieb–Schultz–Mattis Theorem in Open Quantum Systems

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Kyoto, Jun 2024

Recent Developments and Challenges in Topological Phases

Contents

- Reviving the Lieb–Schultz–Mattis Theorem in Open Quantum Systems
  Yi-Neng Zhou, Xingyu Li, Hui Zhai, **CL**, and Yingfei Gu,
  arXiv:2310.01475

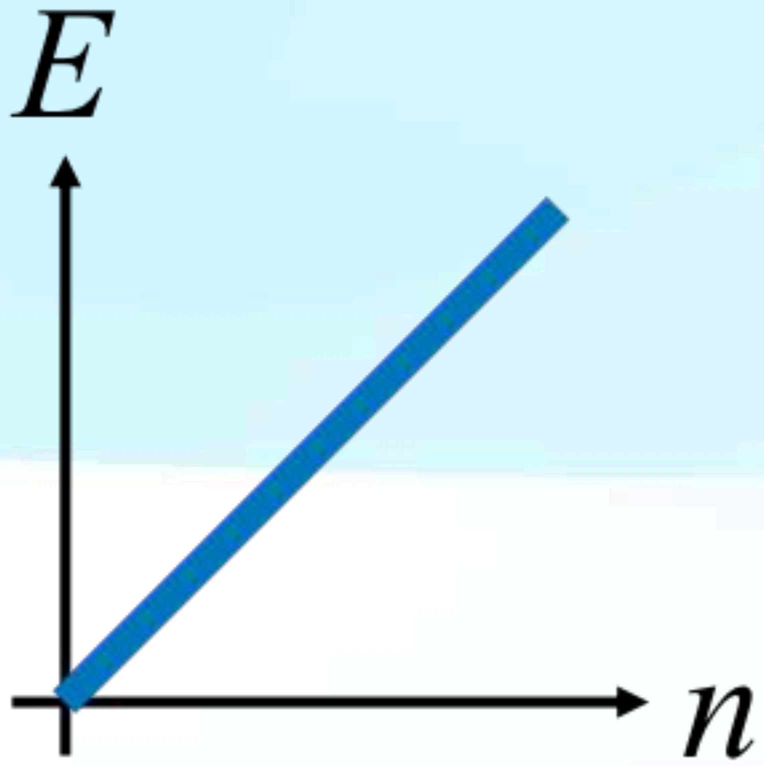
- Numerical investigations of the extensive entanglement Hamiltonian in quantum spin ladders
  CL, Xingyu Li, and Yi-Neng Zhou,
  Quantum Front. **3**, 9 (2024)

A review of the original Lieb–Schultz–Mattis theorem

The Lieb–Schultz–Mattis theorem

The original version

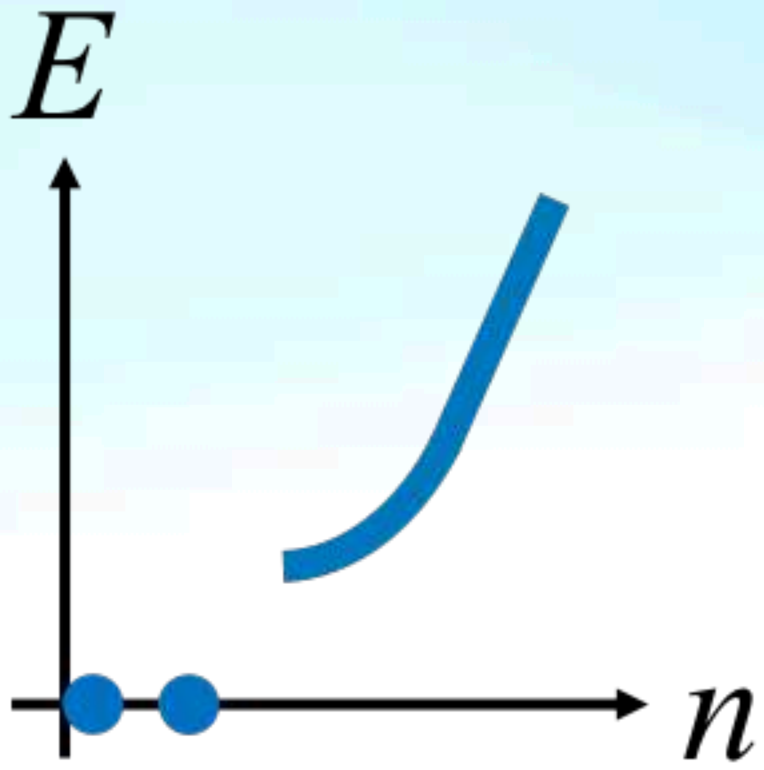
- A spin-1/2, rotational and translational symmetric chain can not be trivially gapped



gapless ✔

e.g. the AFM Heisenberg model

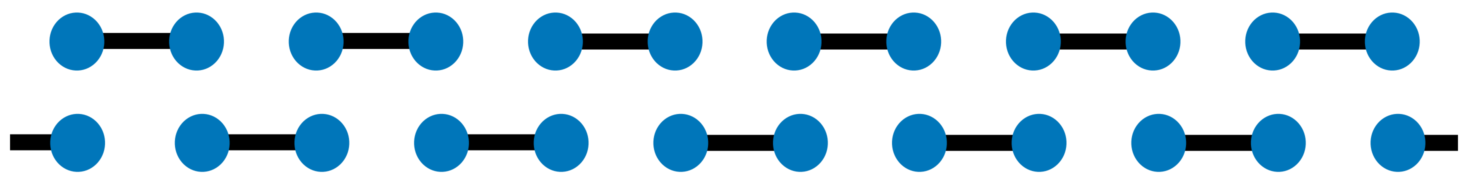
$$H = \sum \mathbf{S}_i \cdot \mathbf{S}_{i+1}$$

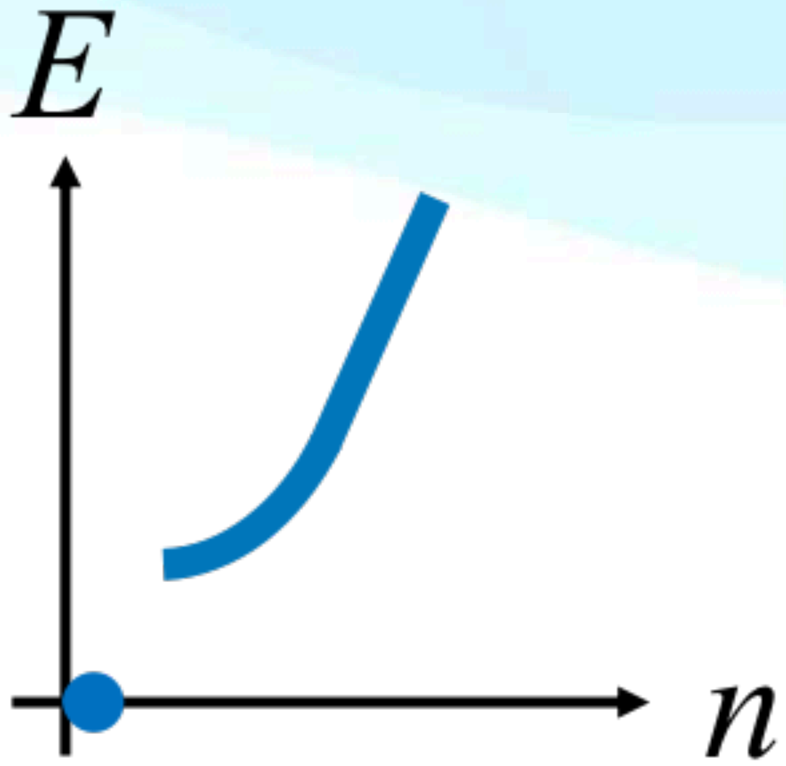


degenerate (SSB) ✔

e.g. the Majumdar–Ghosh model

$$H = \sum \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{1}{2}\mathbf{S}_i \cdot \mathbf{S}_{i+2}$$

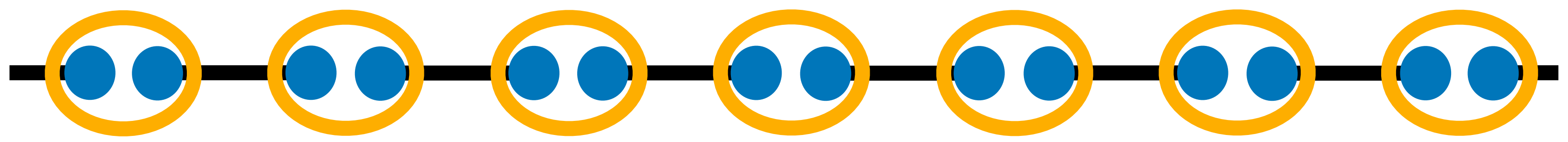




gapped, non degenerate ✗

but spin-1 ✔, e.g. the AKLT model

$$H = \sum \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{1}{3}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2$$



E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961)

Why is LSM important?

UV–IR correspondence

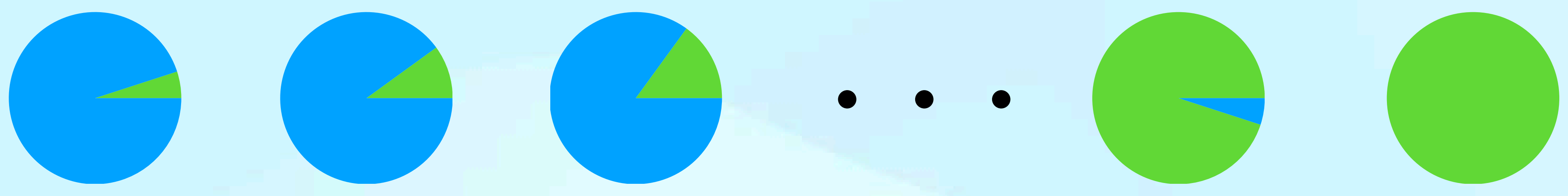
- Input: symmetry, onsite Hilbert space → UV data
- Output: spectrum gap (hence correlation functions) → IR data
- “LSM anomaly”
- Precursor of the Haldane conjecture

M. Cheng and N. Seiberg, SciPost Phys. **15**, 051 (2023)

Sketch of proof of original LSM

Key construction: the twist operator

$$U_{\text{twist}} = \exp\left(\frac{2\pi i}{L}\sum_n n S_z^n\right)$$



- Assume a unique ground state $|\psi\rangle$
  - $|\psi\rangle$ must have spin 0, T eigenvalue e^{ik}
- Now consider $|\phi\rangle = U_{\text{twist}}|\psi\rangle$
  - U twists the state by increasingly large angles on each site
  - key point: U has small effect on operators (Hamiltonian terms and therefore energy), but "changes by a minus sign across the boundary"
  - more precisely, $TU_{\text{twist}}|\psi\rangle = -U_{\text{twist}}T|\psi\rangle = -e^{ik}U_{\text{twist}}|\psi\rangle$
  - hence $|\phi\rangle$ has energy close to $|\psi\rangle$, but $\langle\psi|\phi\rangle = 0$

E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961)

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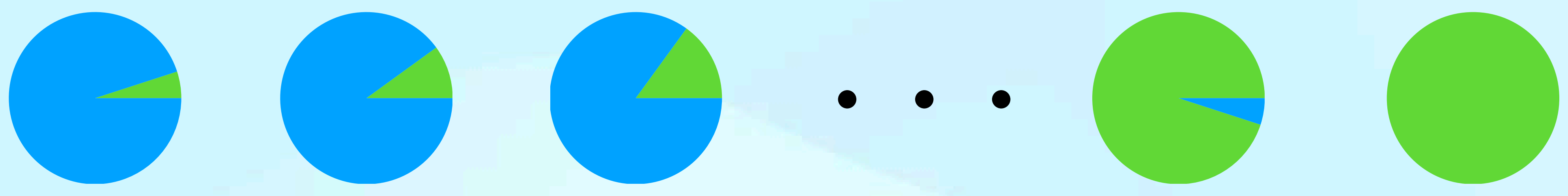

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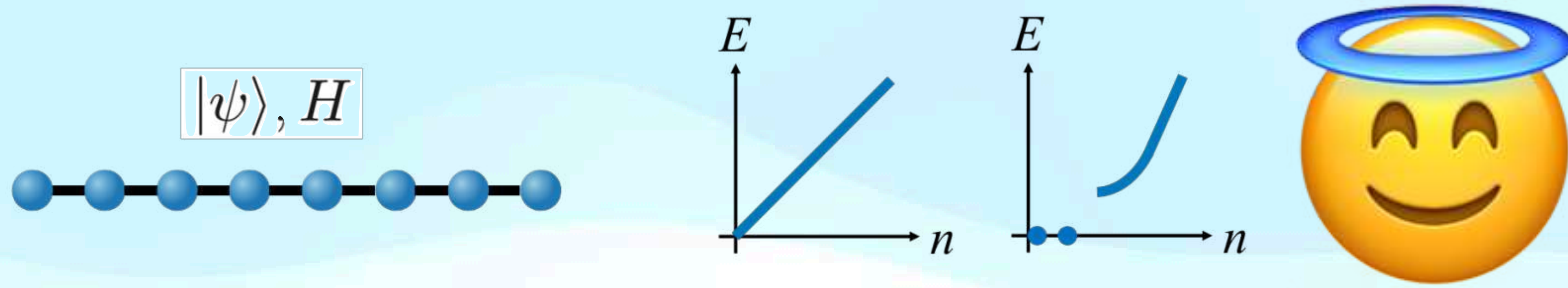
E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **16**, 407 (1961)

What happens to LSM in open systems?

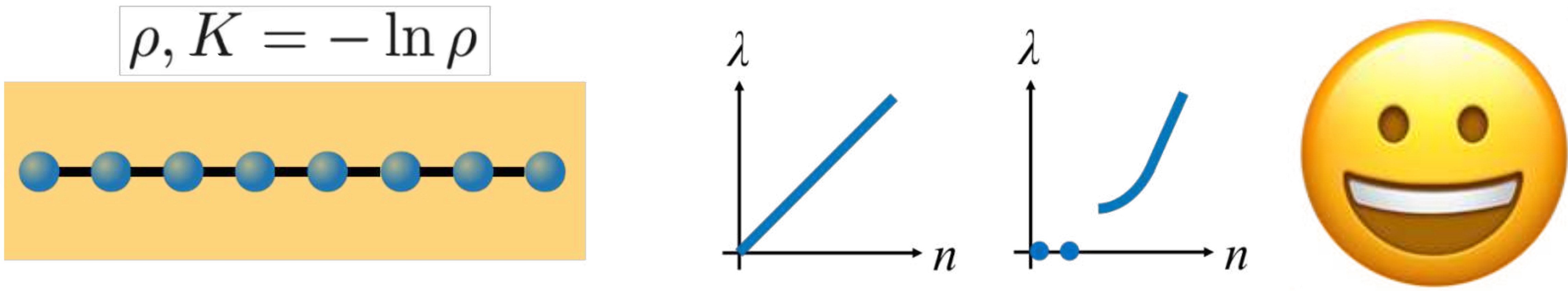
Reviving LSM in open systems

- When coupled to a bath, energy & spectrum gap no longer well defined



- Correlations can become short-range

- Is there a way to revive the LSM?

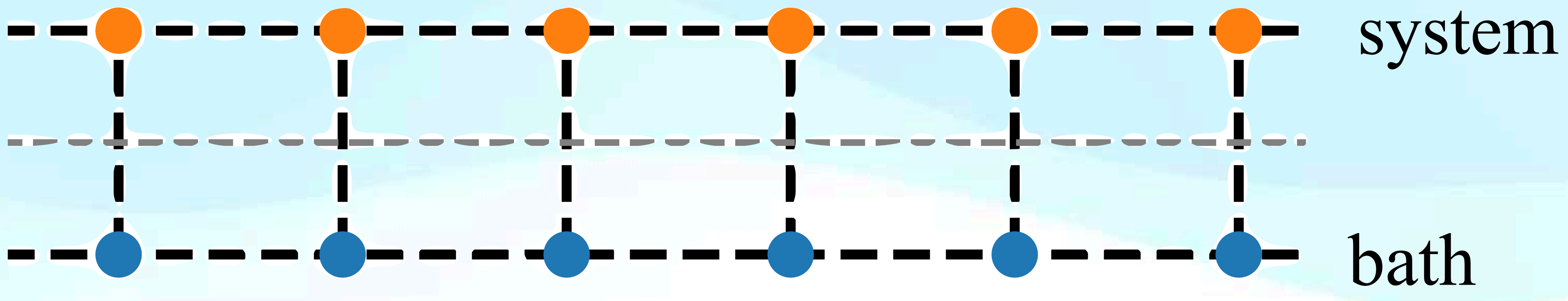


- Idea: use density matrix ρ & entanglement Hamiltonian $K = -\ln \rho$

Y.-N. Zhou, X. Li, H. Zhai, **CL**, and Y. Gu, arXiv:2310.01475

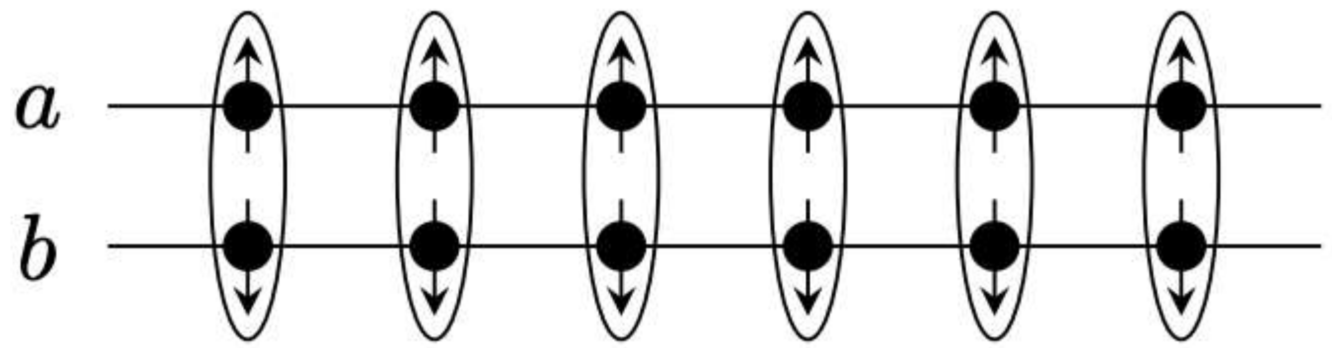
Some intuitions of entanglement LSM

- Trivial example: heat bath, $K \propto H$



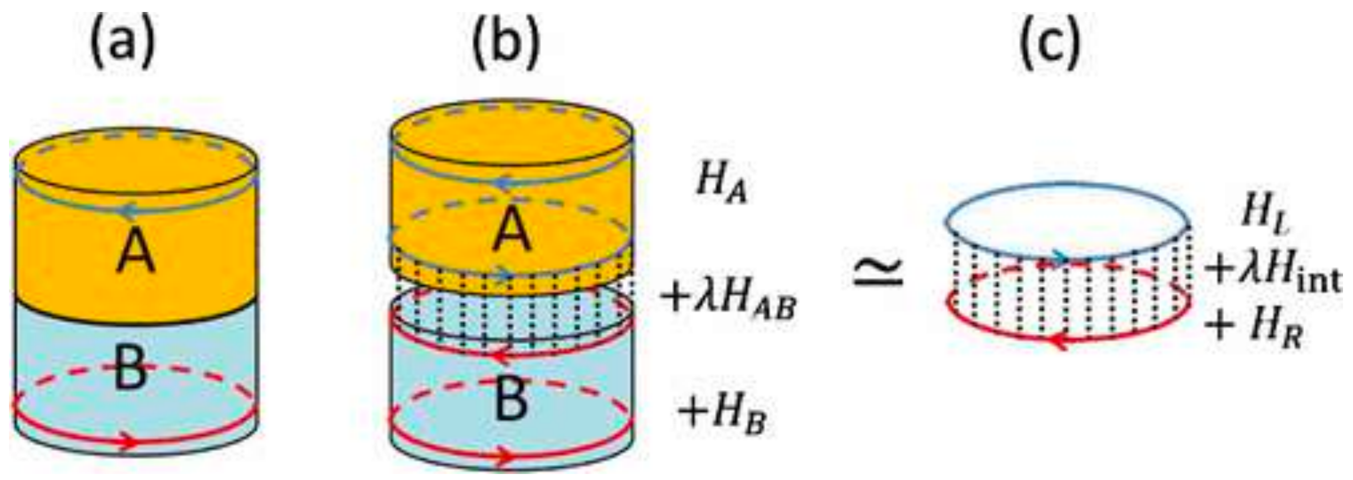
- Coupled chain setup
  - Strong coupling limit, perturbative wavefunction

$$|\psi\rangle \propto |0\rangle - \frac{1}{2\Delta}(H_a + H_b)|0\rangle \approx e^{-\beta(H_a+H_b)/2}|0\rangle$$



  - Weak coupling limit, Qi–Katsura–Ludwig construction

$$K \sim H$$



X.-L. Qi, H. Katsura, and A. W. W. Ludwig, Phys. Rev. Lett. **108**, 196402 (2012)

Formulation of entanglement LSM

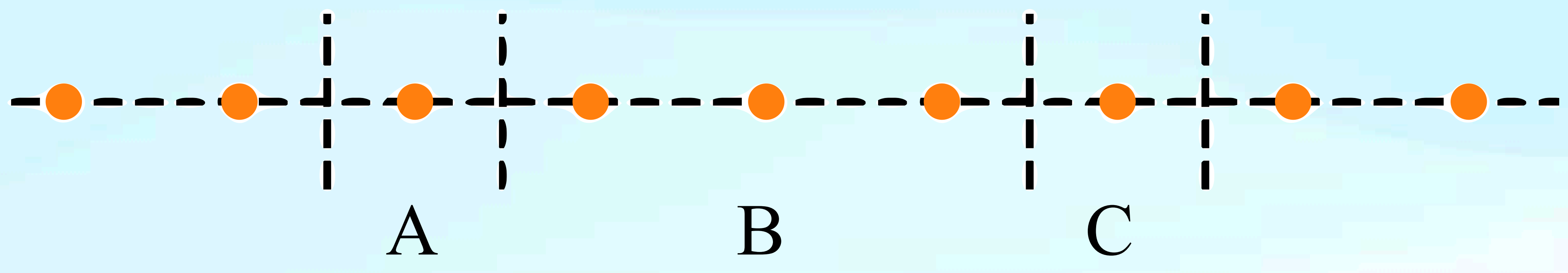
- Half integer spin
- Weak symmetry, i.e. $U^\dagger \rho U = \rho \Leftrightarrow U^\dagger K U = K$, $U = \mathrm{rotation}, \mathrm{translation}$
  - satisfied in the coupled chain setup if the total system has the symmetry
- Short-range correlated, i.e. $\langle O_j O_k \rangle - \langle O_j \rangle \langle O_k \rangle \sim e^{-|j-k|/\xi}$
  - a natural condition if coupling to bath is strong enough
  - necessary to guarantee quasi-locality
- Under these conditions, the proof for original LSM goes through

Y.-N. Zhou, X. Li, H. Zhai, **CL**, and Y. Gu, arXiv:2310.01475

Localness of entanglement Hamiltonian

“Quantum Markov chain has local entanglement Hamiltonian”

- quantum conditional mutual information $I(A:C|B) = S_{AB} + S_{BC} - S_B - S_{ABC}$



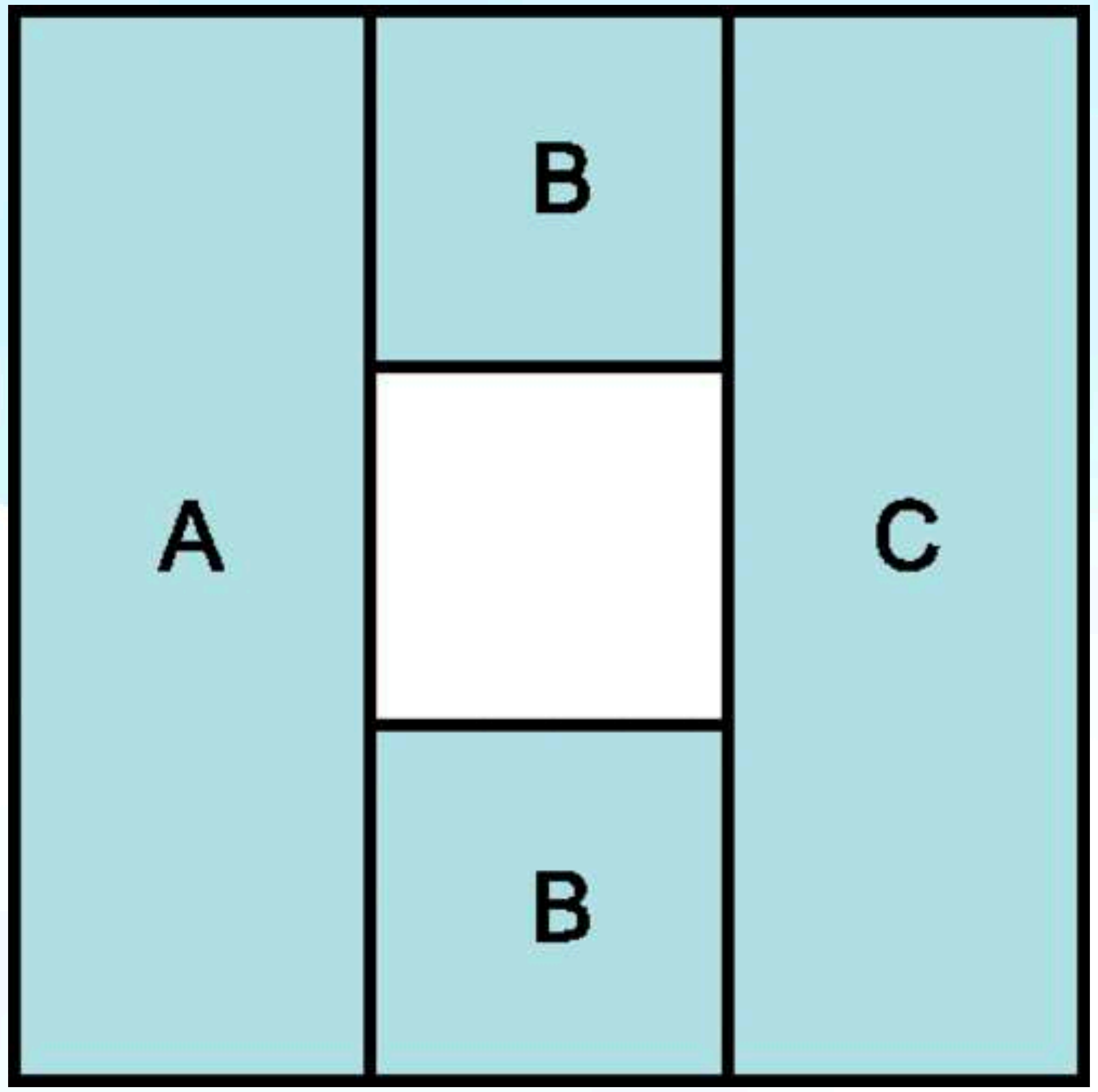
$$I(A:C|B) = 0 \Rightarrow \text{ local } K$$

short-range correlated $\Rightarrow I(A:C|B) < \varepsilon \Rightarrow$ quasi-local K

D. Petz, Rev. Math. Phys. **15**, 79 (2003).

K. Kato and F. G. S. L. Brandão, Commun. Math. Phys. **370**, 117 (2019)

Aside: QCMI and topo. entanglement entropy



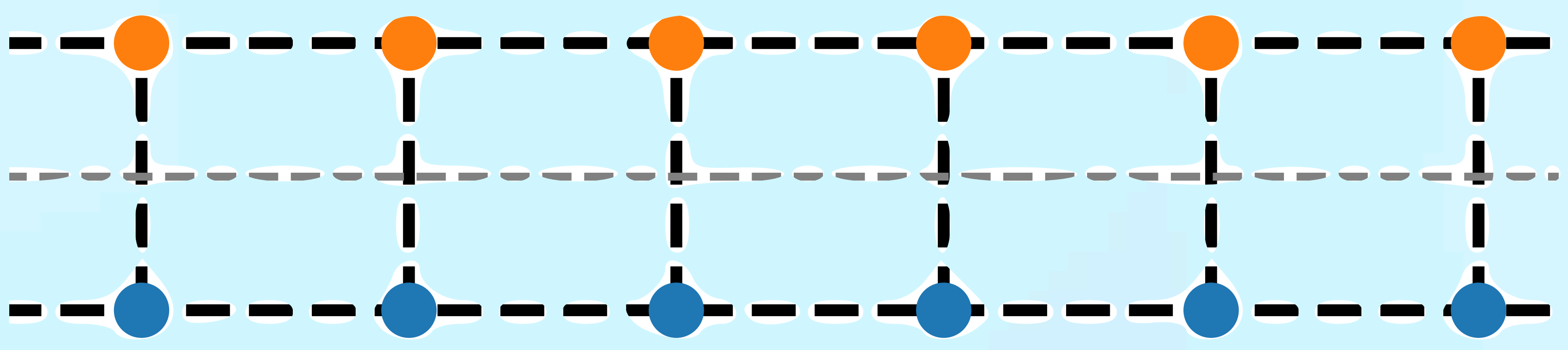
$$I(A:C|B) = S_{AB} + S_{BC} - S_B - S_{ABC} = 2\gamma$$

A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006)
M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006)

Numerical results

Numerical example I

An AKLT ladder

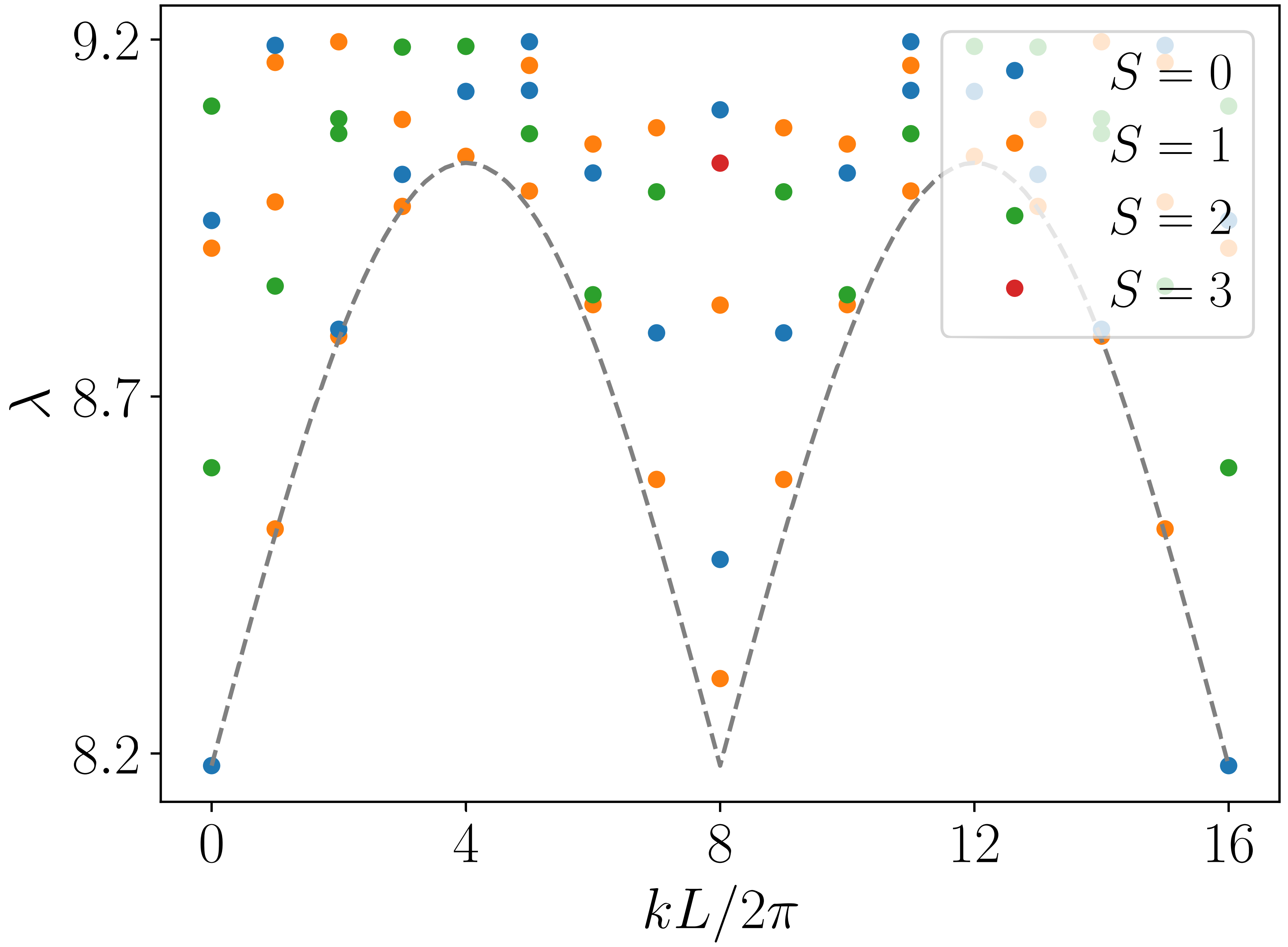


- A spin-1/2 chain coupled to a spin-3/2 chain

$$H = \sum_{i=1}^{L} J_1 \left(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{1}{3}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2\right) + J_2 \mathbf{S}_{i,s} \cdot \mathbf{S}_{i,b}, \ \mathbf{S}_i = \mathbf{S}_{i,s} + \mathbf{S}_{i,b}$$

- The ground state is exactly known and trivially gapped

- The entanglement spectrum is very similar to the energy spectrum of the Heisenberg model

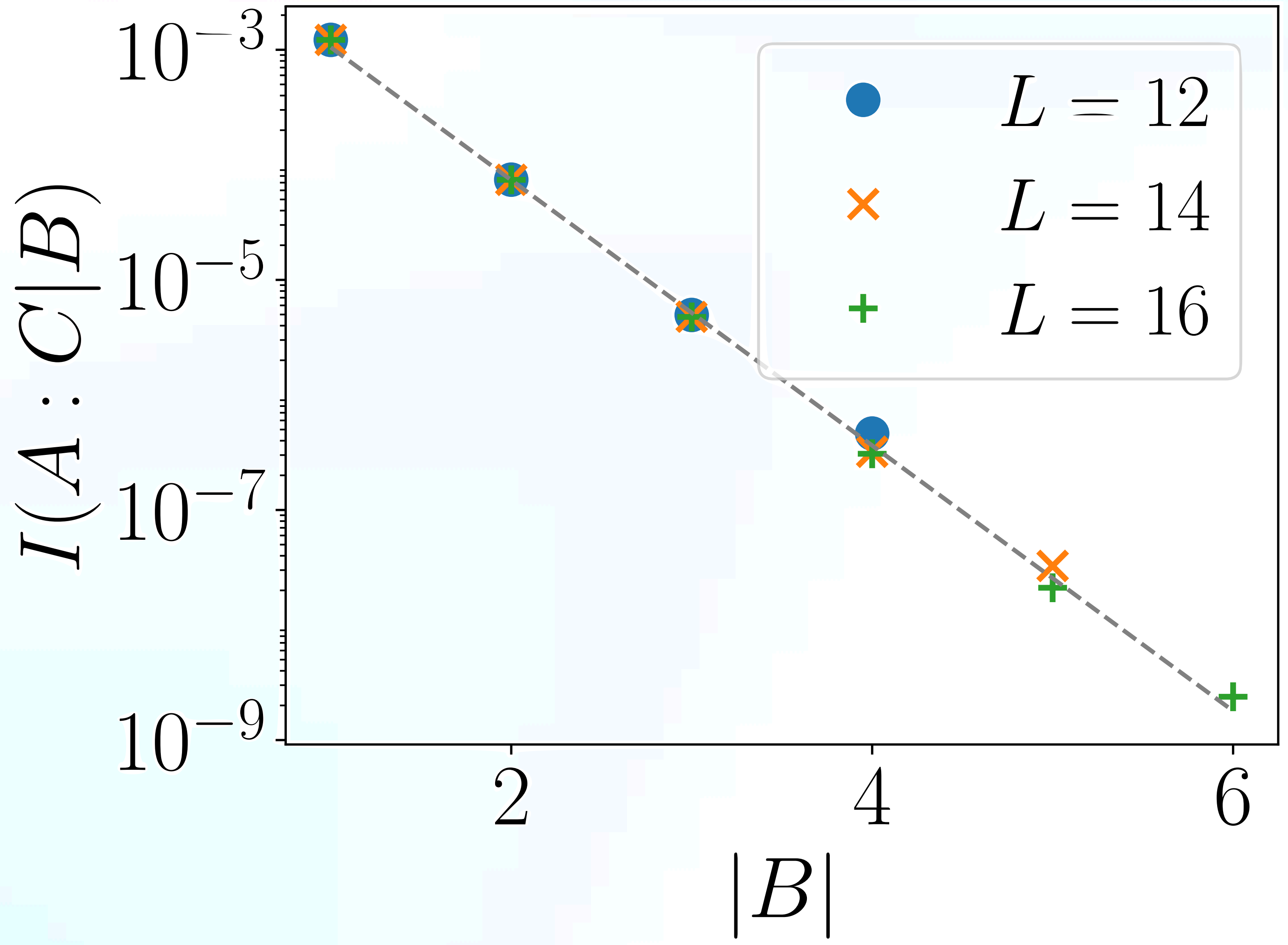


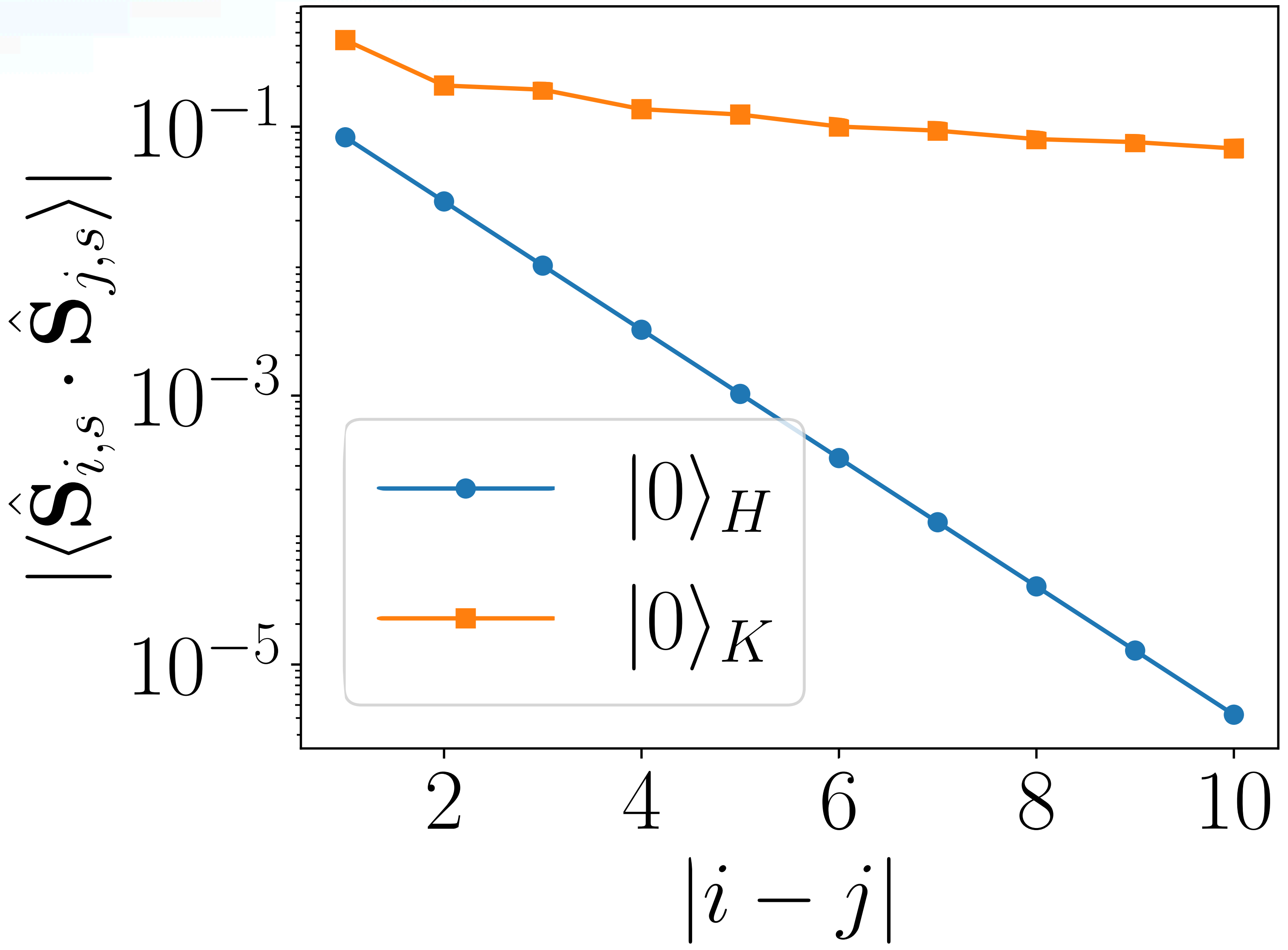
Y.-N. Zhou, X. Li, H. Zhai, **CL**, and Y. Gu, arXiv:2310.01475

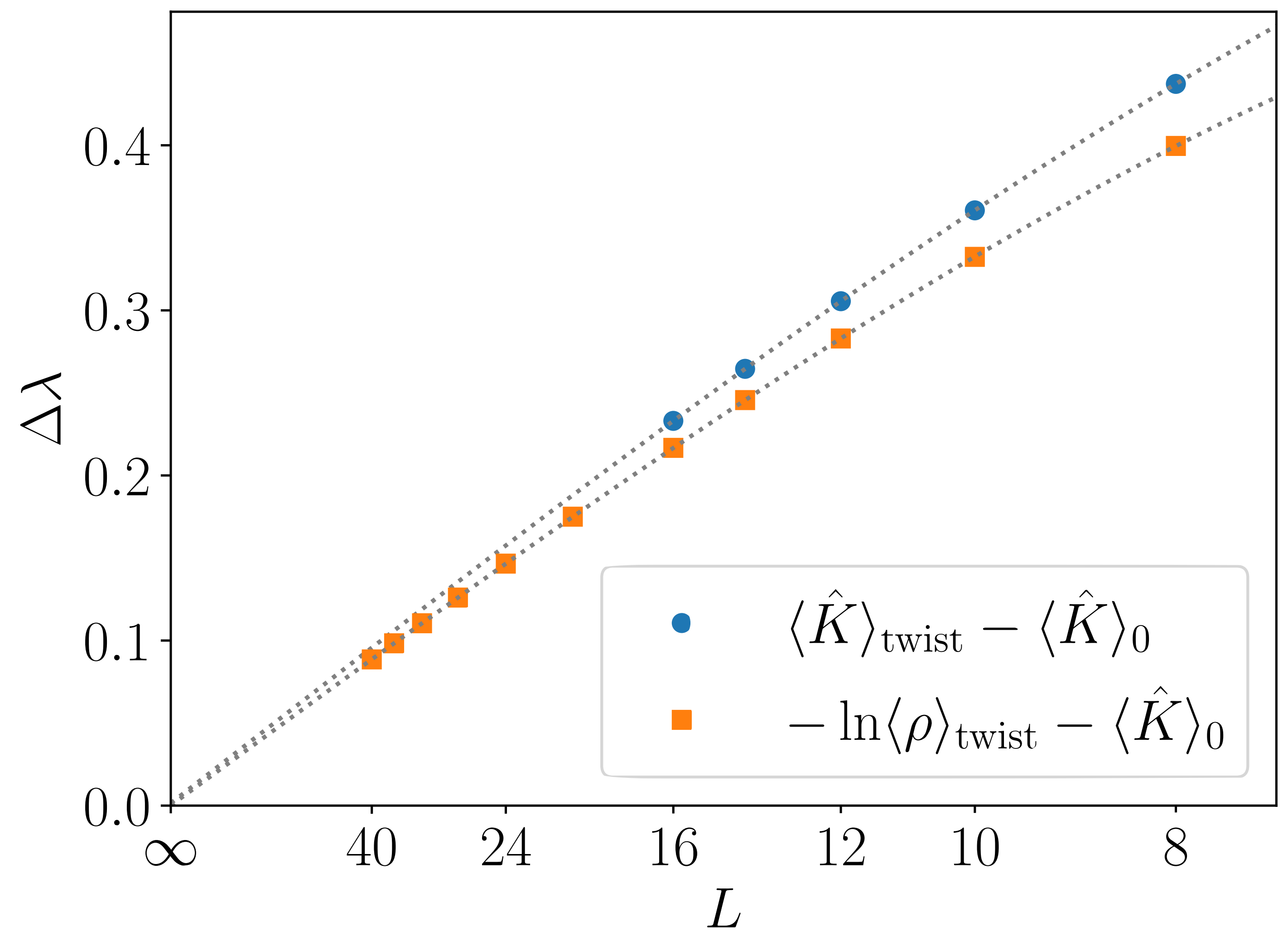
Numerical example I

An AKLT ladder

- Decay of quantum conditional mutual information ✔
- Slow decay of correlation function ✔
- Vanishing of energy difference between ground state and twisted state ✔







Y.-N. Zhou, X. Li, H. Zhai, **CL**, and Y. Gu, arXiv:2310.01475

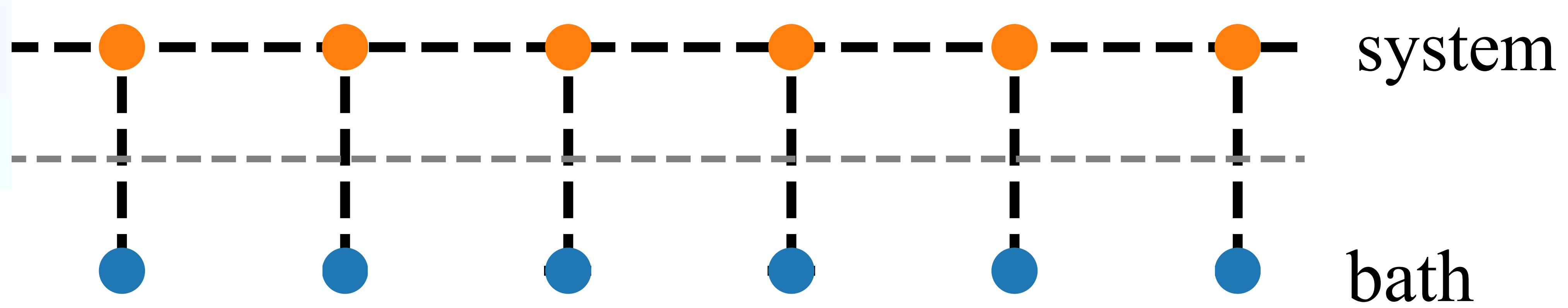
Numerical example II

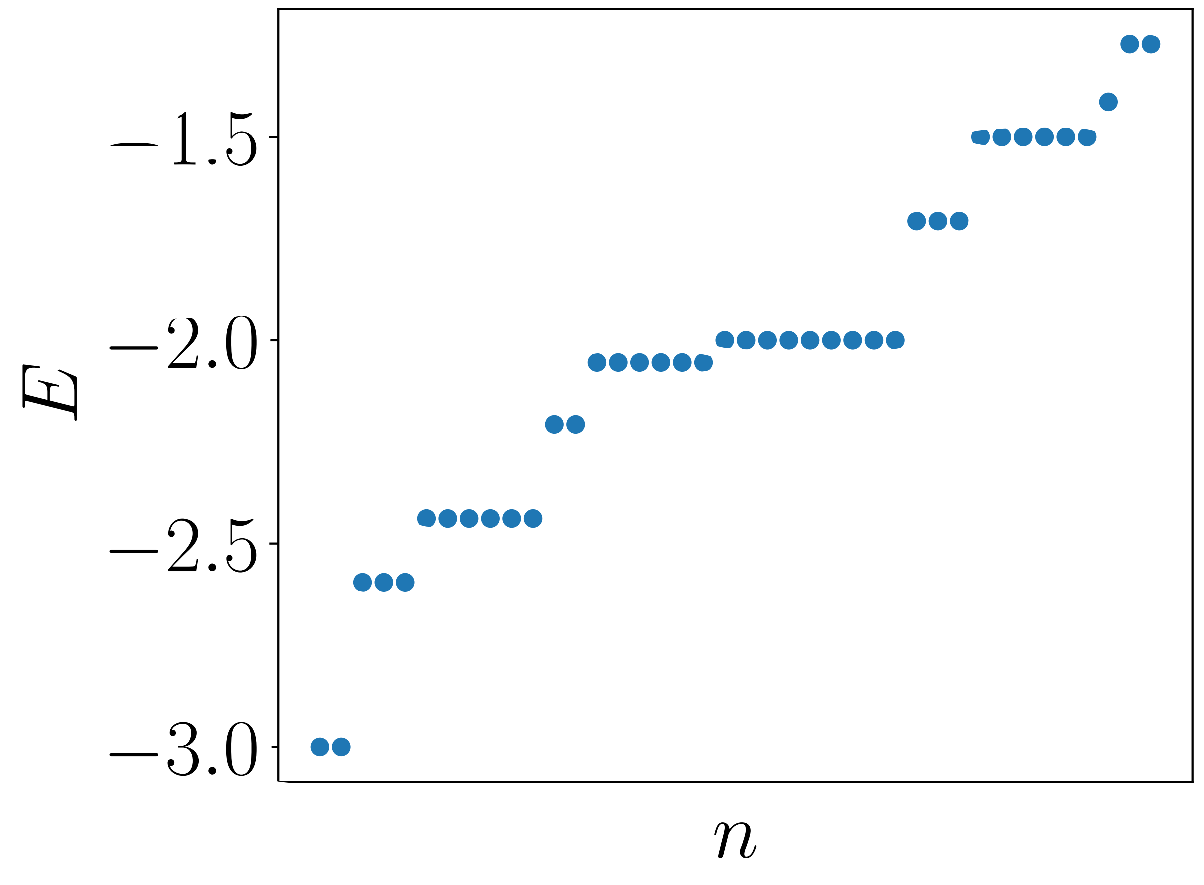
Decohered Majumdar–Ghosh ladder

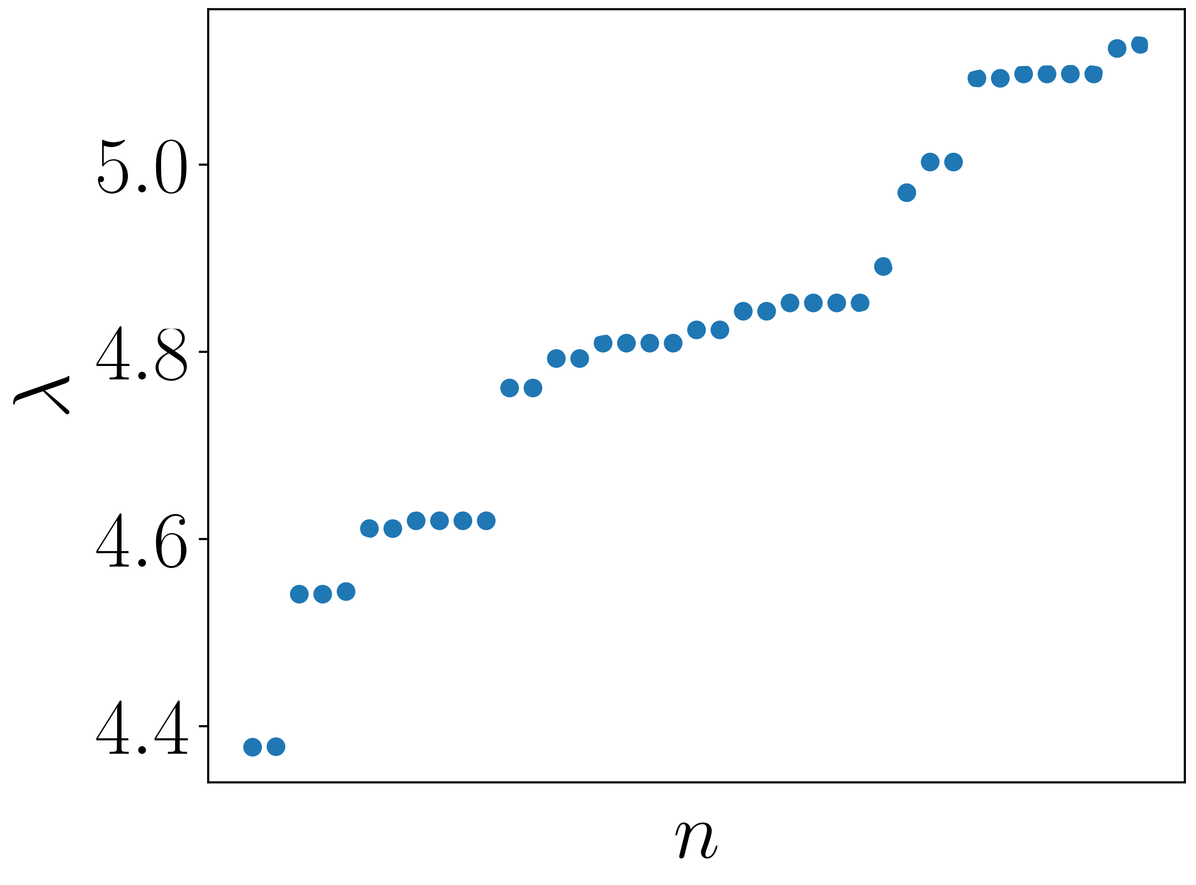
- We decohere the Majumdar–Ghosh chain by coupling it to spin-3/2 modes

$$H = \sum_{i=1}^{L} J_1(\mathbf{S}_{i,s} \cdot \mathbf{S}_{i+1,s} + \frac{1}{2}\mathbf{S}_{i,s} \cdot \mathbf{S}_{i+2,s}) + J_2 \mathbf{S}_{i,s} \cdot \mathbf{S}_{i,b} + D(S^z_{i,s} + S^z_{i,b})^2$$

- The total system is trivially gapped, while one expects SSB in entanglement spectrum, similar to original MG







Y.-N. Zhou, X. Li, H. Zhai, **CL**, and Y. Gu, arXiv:2310.01475

More numerical investigations

Two AKLT ladders

- We consider both spin-1/2 & spin-3/2 baths
- Use DMRG to obtain results for large system size
- Fitting against CFT formulae

$$\frac{\lambda_0(L)}{L} = \lambda_\infty - \frac{\pi c v}{6L^2} + \cdots$$

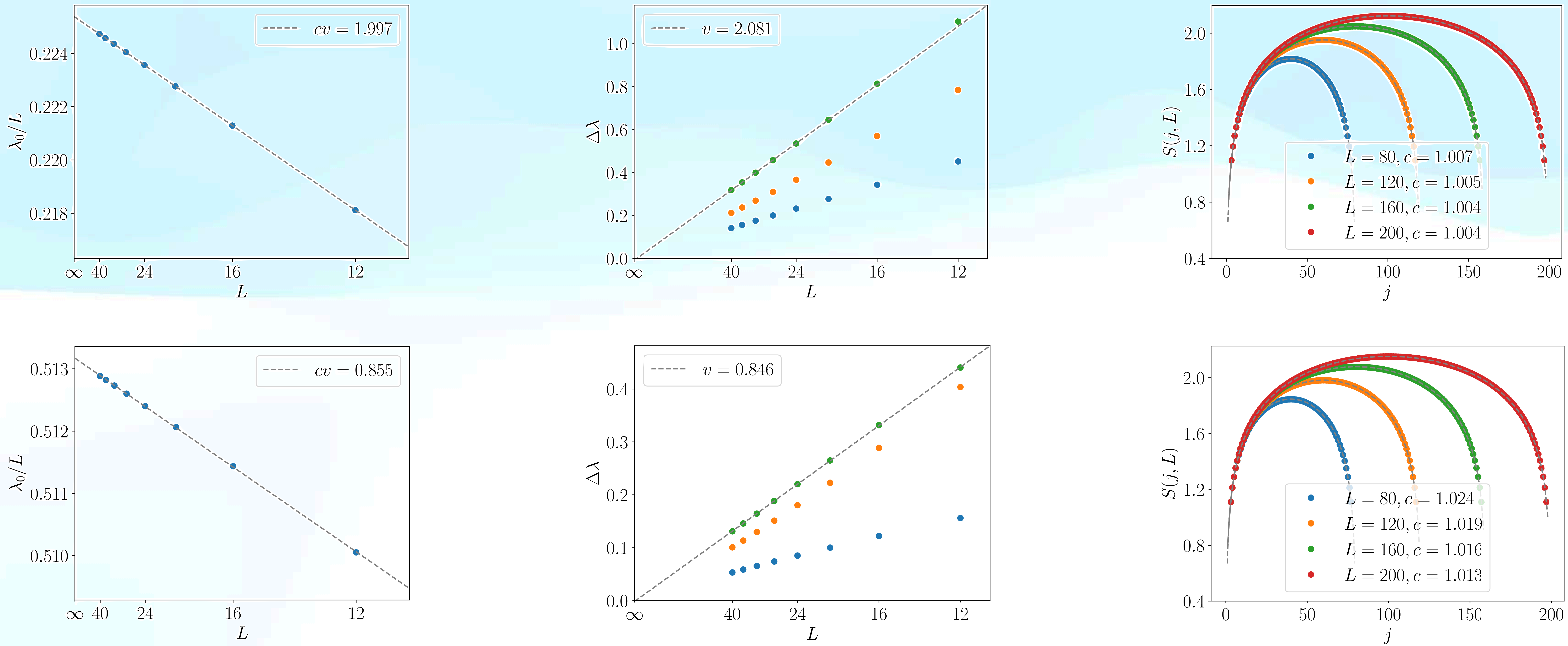
$$\Delta\lambda = \frac{2\pi v}{L} + \cdots$$

$$S(j, L) = \frac{c}{3} \ln \left[\frac{L}{\pi} \sin \left(\frac{\pi j}{L} \right) \right] + S_0$$

CL, X. Li, and Y.-N. Zhou, Quantum Front. **3**, 9 (2024)

More numerical investigations

Two AKLT ladders

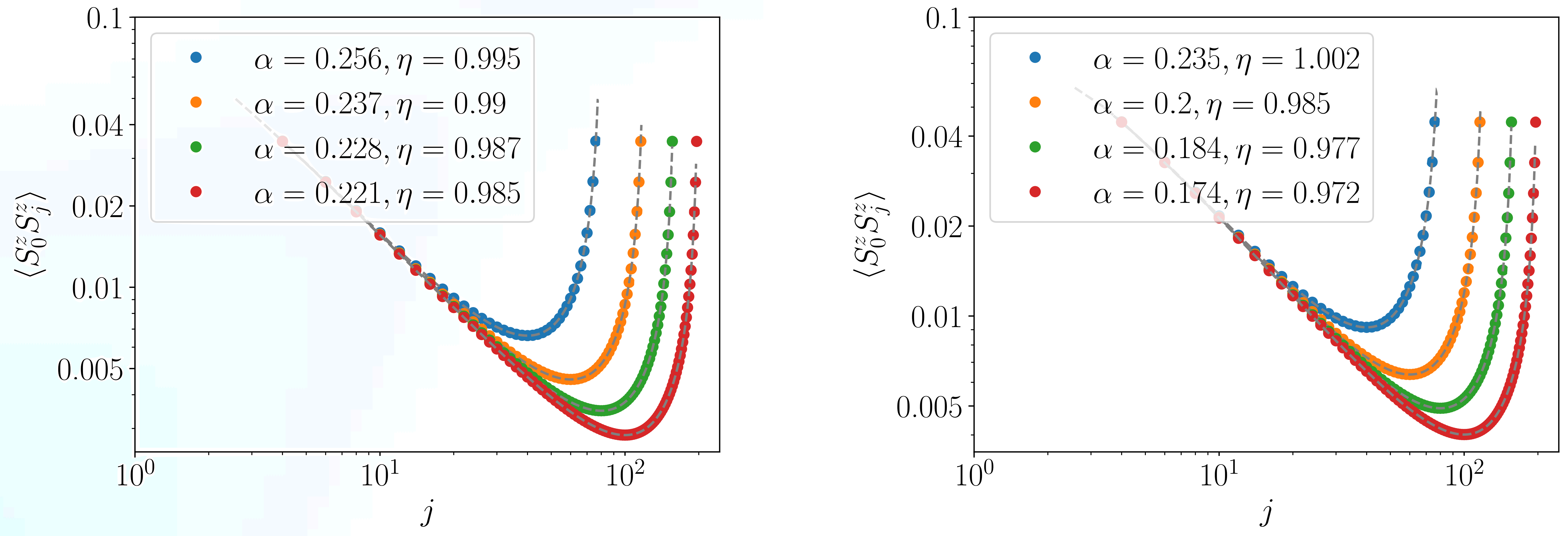


CL, X. Li, and Y.-N. Zhou, Quantum Front. **3**, 9 (2024)

More numerical investigations

Two AKLT ladders

- We can also fit the correlation functions with $\langle S_0^z S_j^z \rangle \propto \frac{(\ln(c\tilde{j}))^\alpha}{\tilde{j}^\eta}, \ \tilde{j} = \sin(\pi j/L)$
- All the results are consistent with $\mathrm{SU}(2)_1$ WZW CFT
- In the spin-3/2 case, we get the CFT for a spin-3/2 model for free

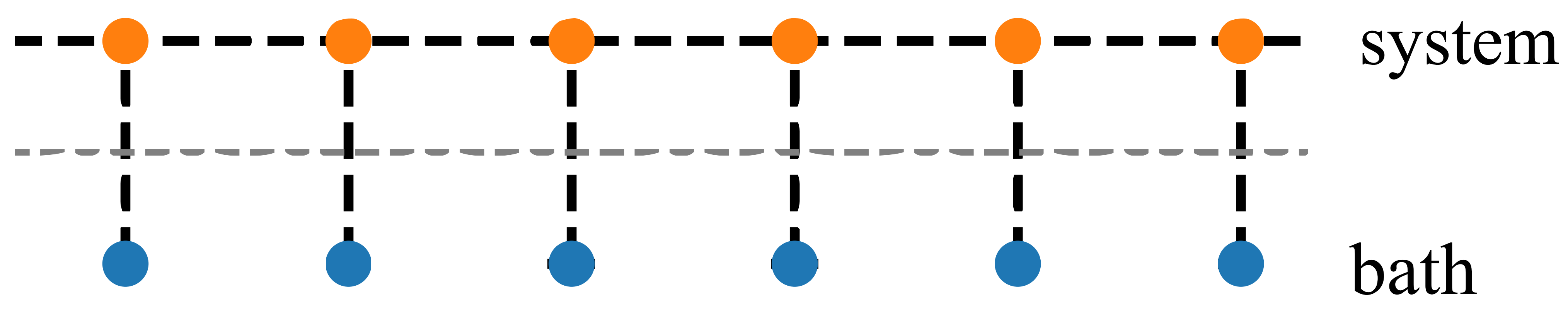


CL, X. Li, and Y.-N. Zhou, Quantum Front. **3**, 9 (2024)

More numerical investigations

Two decohered MG ladders

- Again, we consider both spin-1/2 & spin-3/2 baths
- Need to do DMRG twice: once to get the ground state, once on the reduced density matrix

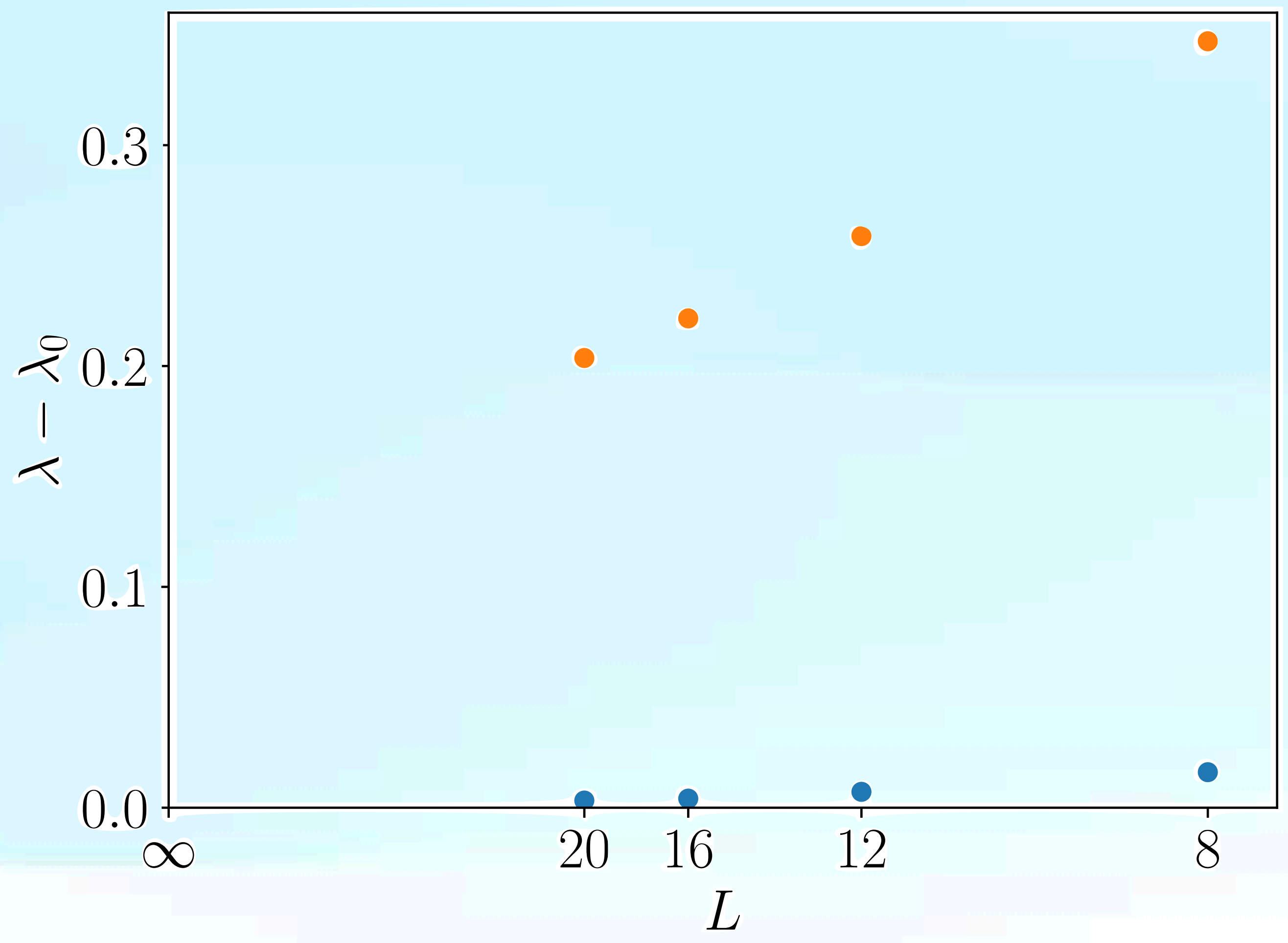


CL, X. Li, and Y.-N. Zhou, Quantum Front. **3**, 9 (2024)

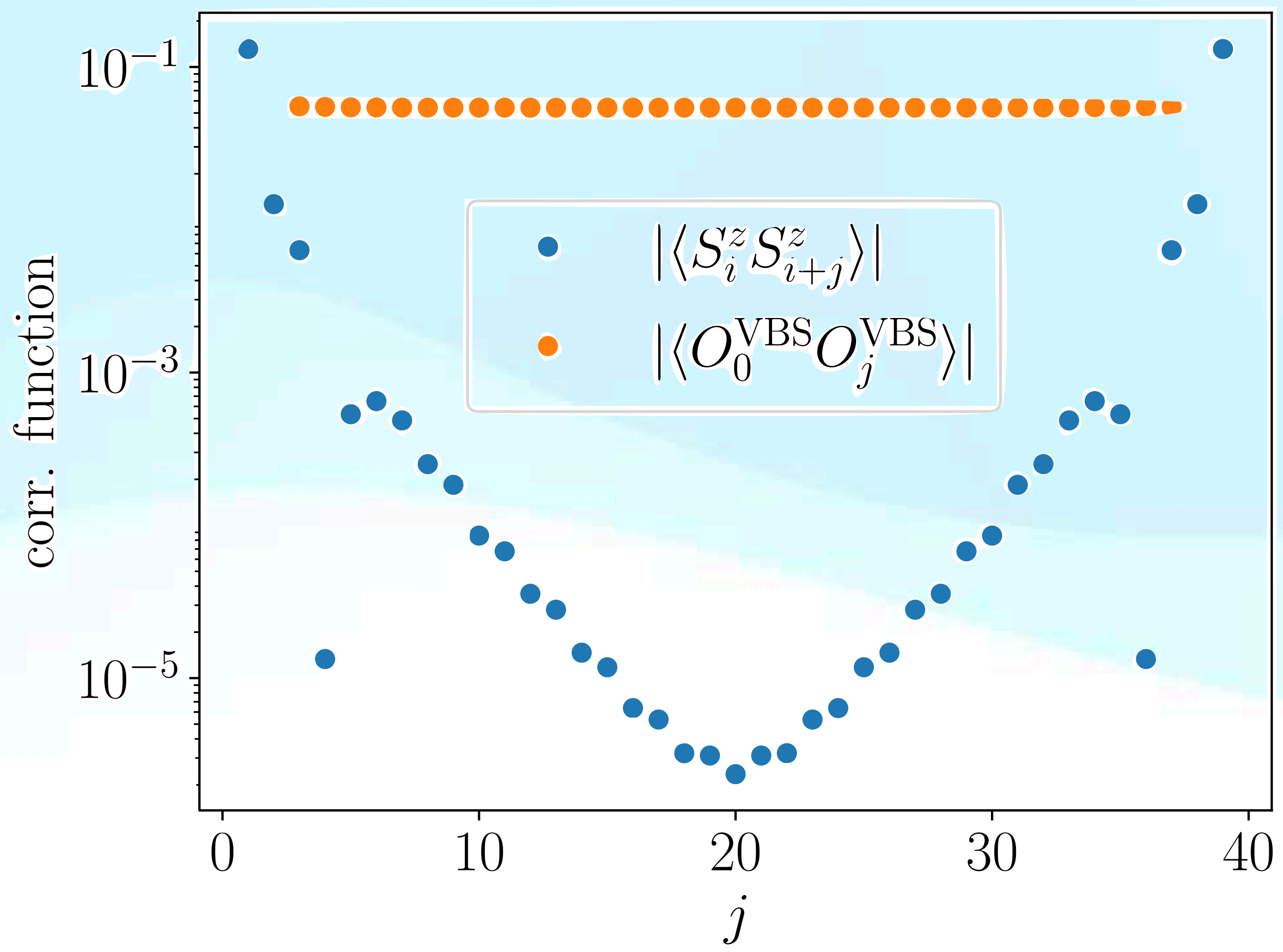
More numerical investigations

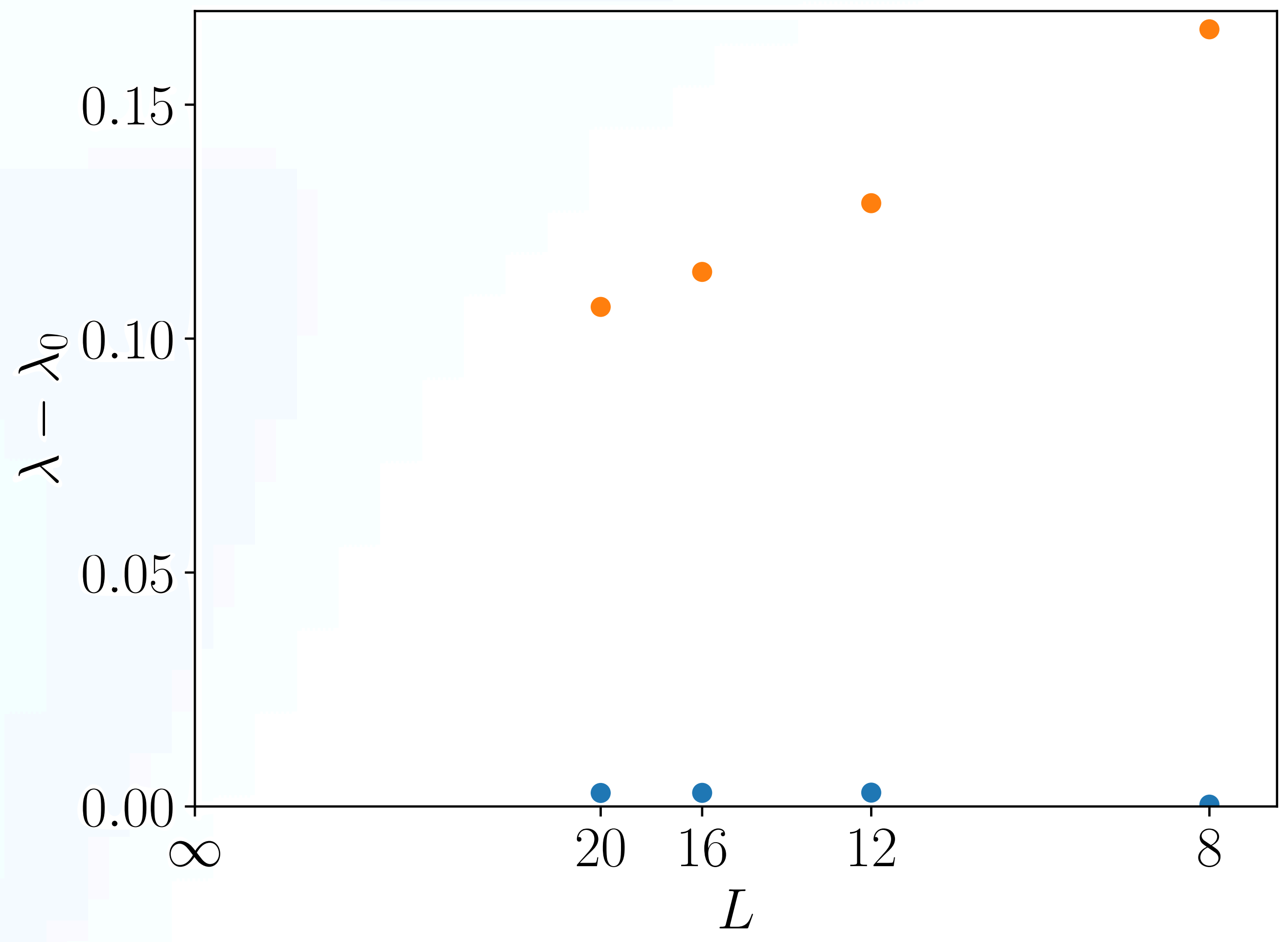
Two decohered MG ladders

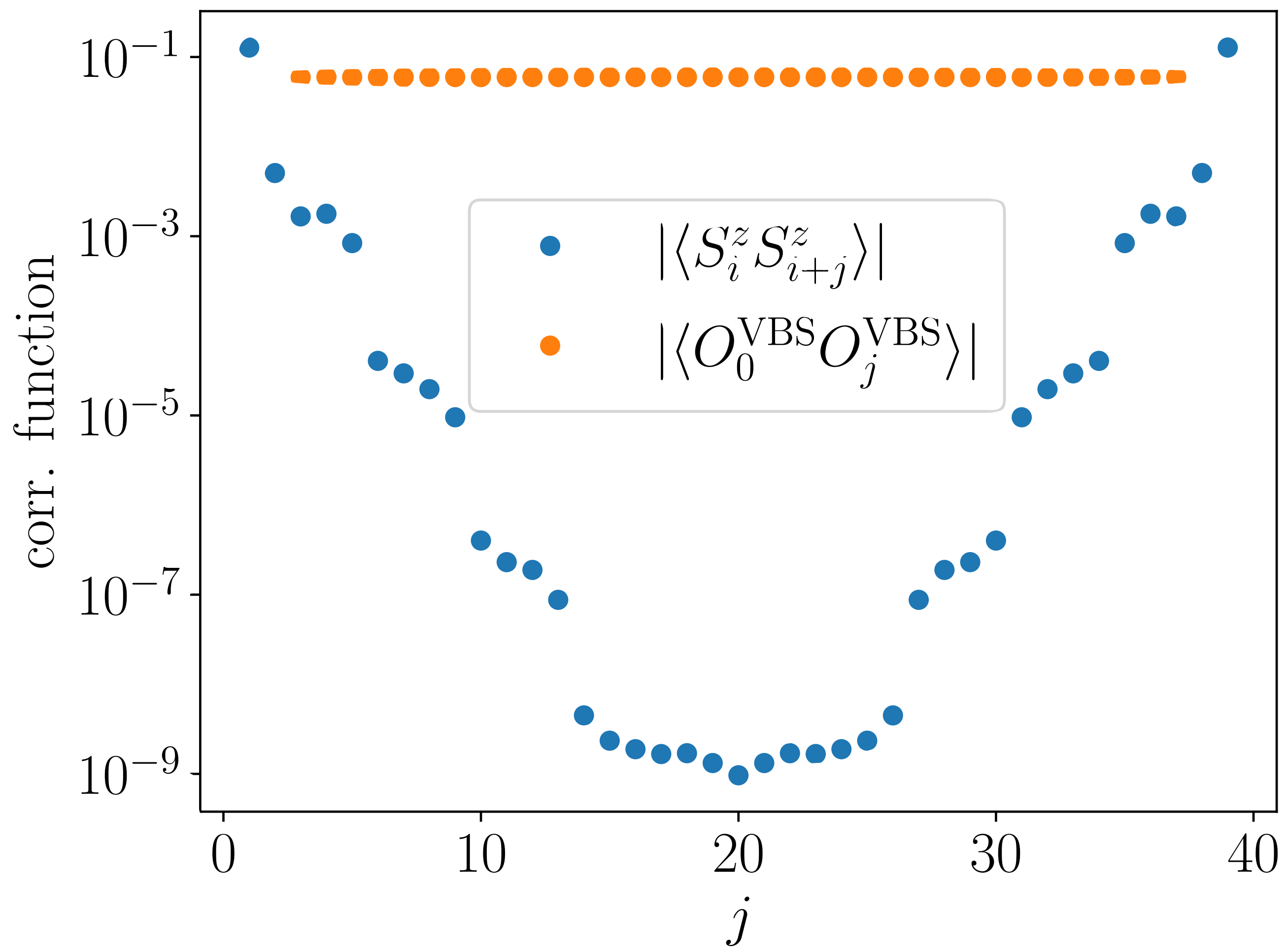
gap



corr. func.







CL, X. Li, and Y.-N. Zhou, Quantum Front. **3**, 9 (2024)

Summary

- We generalize the LSM theorem to open systems, focusing on the entanglement Hamiltonian
- Original symmetry conditions → weak symmetry
- Double identity of short-range correlation
- Extensive numerical investigations performed to corroborate the proposal

Further thoughts

- There are mulitple approaches to “openization”
  - LSM in Lindbladian [K. Kawabata, R. Sohal, and S. Ryu, Phys. Rev. Lett. **132**, 070402 (2024)]
  - multiple facets of open systems
- Tomita–Takesaki theory and modular flow $\rho^{is} = e^{-iKs}$
- “Entanglement bootstrap”

E. Witten, Rev. Mod. Phys. **90**, 045003 (2018)

works by Kim, Shi, Kato, Albert, McGreevy et al.

Collaborators

- Yi-Neng Zhou (IASTU→Geneva)
- Xingyu Li (IASTU)
- Hui Zhai (IASTU)
- Yingfei Gu (IASTU)




Acknowledgements

- Zhen Bi (Penn state)
- Meng Cheng (Yale)
- Chao-Ming Jian (Cornell)
- Shenghan Jiang (KITS)
- Chong Wang (PI)
- Chao Yin (UC Boulder)

Thank you!